Game Physics

Game and Media Technology
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Soft body physics

Soft bodies

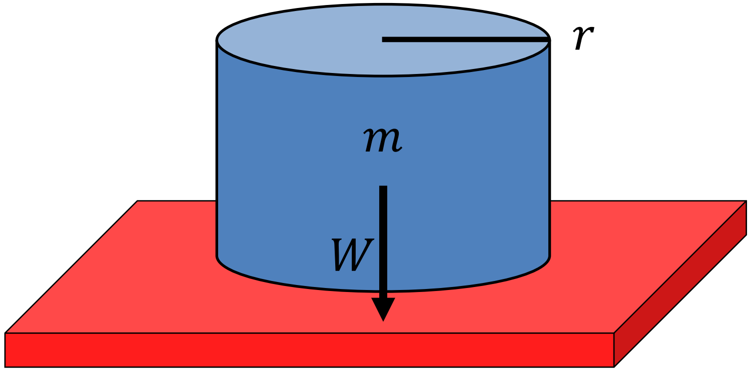
- In reality, objects are not purely rigid
  - for some it is a good approximation
  - but if you hit them with enough force, they will deform or break down
- In a game, you often want to see soft bodies (*i.e.* deformable objects)
  - car body, anything you punch or shoot at, *etc.*
  - piece of cloth, flag, paper sheet, *etc.*
  - snow, mud, lava, liquid, *etc.*

Elasticity

- Elasticity is the primary concept in soft body physics
- Property by which the body returns to its original shape after the forces causing the deformation are removed
  - A plastic rod can easily be bended, and returned to its original form
  - A steel rod is difficult to bend, but can also return to its original form

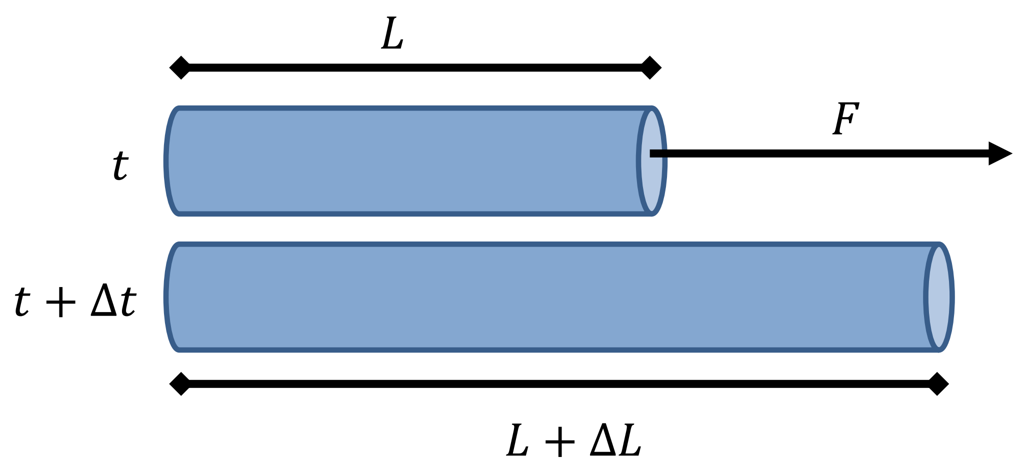
Stress

- The stress within an object is the magnitude of an applied force divided by the area of its application
  - large value when the force is large or when the surface is small
- It is a pressure measure σ and has the unit Pascal $Pa = N/m^2$
- Example
  - the stress on the plane is $\sigma = mg/(\pi r^2)$



Strain

- The strain on an object ϵ is the fractional deformation caused by a stress
  - dimensionless (change in dimension relative to original dimension)
  - measures how much a deformation differs from a rigid body transformation
    - negative if compression, zero if rigid body transformation, positive if stretch
- Example
  - the strain on the rod is $\epsilon = \Delta L / L$



Body material

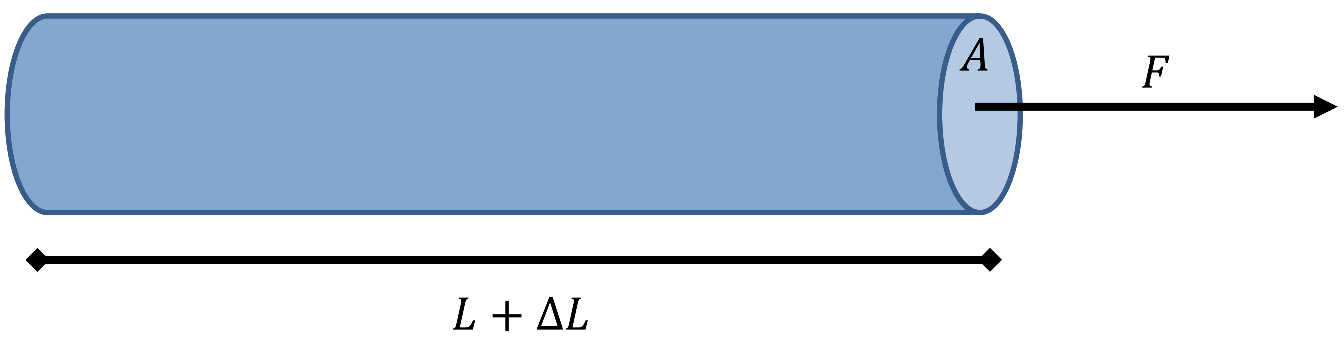
- Stress and strain do not contain information about the specific material (*i.e.* deformation behavior) to which a force is applied
- The amount of stress to produce a strain does
- Therefore we can model it by the ratio of stress to strain
  - usually in a linear direction, along a planar region or throughout a volume region
    - Young’s modulus, Shear modulus, Bulk modulus
  - they describe the different ways the material changes shape due to stress

Young's modulus

- The Young's modulus is defined as the ratio of linear stress to linear strain

$$Y = \frac{linear\ stress}{linear\ strain} = \frac{F/A}{\Delta L/L}$$

- Example

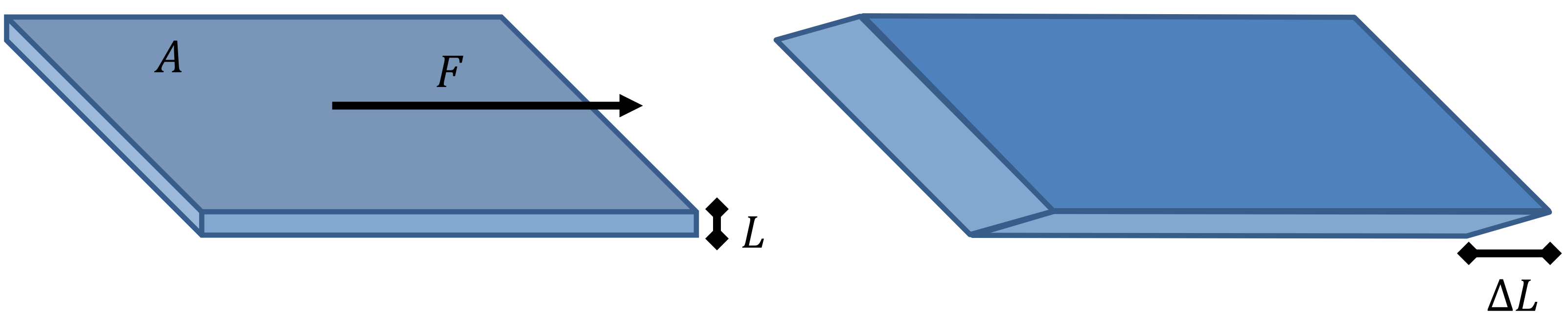


Shear modulus

- The Shear modulus is defined as the ratio of planar stress to planar strain

$$S = \frac{planar\ stress}{planar\ strain} = \frac{F/A}{\Delta L/L}$$

- Example

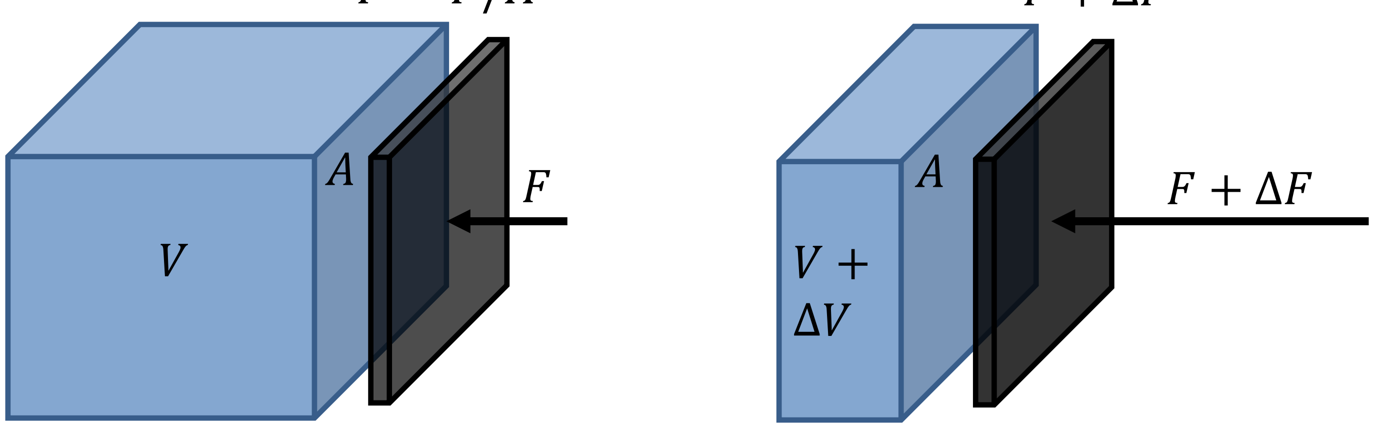


Bulk modulus

- The Bulk modulus is defined as the ratio of volume stress to volume strain (inverse of compressibility)

$$B = \frac{volume\ stress}{volume\ strain} = \frac{\Delta P}{\Delta V / V}$$

- Example



Poisson's ratio

- The Poisson's ratio is the ratio of transverse to axial strain

$$\nu = -\frac{d\ transverse\ strain}{d\ axial\ strain}$$

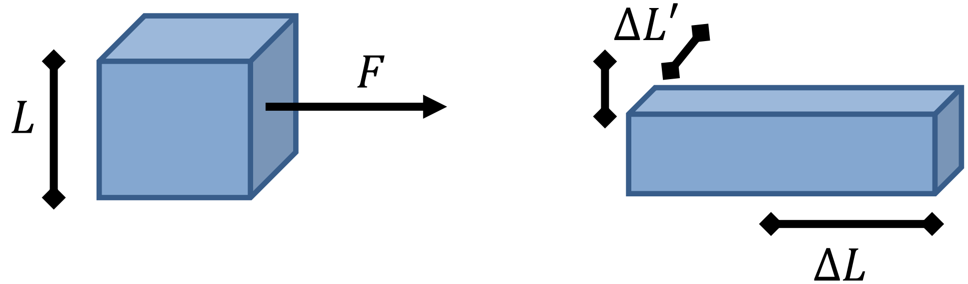
  - negative transverse strain in axial tension, positive in axial compression
  - negative axial strain in compression, positive in tension
  - equals 0.5 in perfectly incompressible material
- If the force is applied along x then we have

$$\nu = -\frac{d\epsilon_y}{d\epsilon_x} = -\frac{d\epsilon_z}{d\epsilon_x}$$

Poisson's ratio

8.1-8.5

- Example of a cube of size L



$$d\epsilon_x = \frac{dx}{x} \quad d\epsilon_y = \frac{dy}{y} \quad d\epsilon_z = \frac{dz}{z}$$

$$-\nu \int_L^{L+\Delta L} \frac{dx}{x} = \int_L^{L-\Delta L'} \frac{dy}{y} = \int_L^{L-\Delta L'} \frac{dz}{z} \Leftrightarrow$$

$$\left(1 + \frac{\Delta L}{L}\right)^{-\nu} = 1 - \frac{\Delta L'}{L} \Leftrightarrow \nu \approx \frac{\Delta L'}{\Delta L}$$

Continuum mechanics

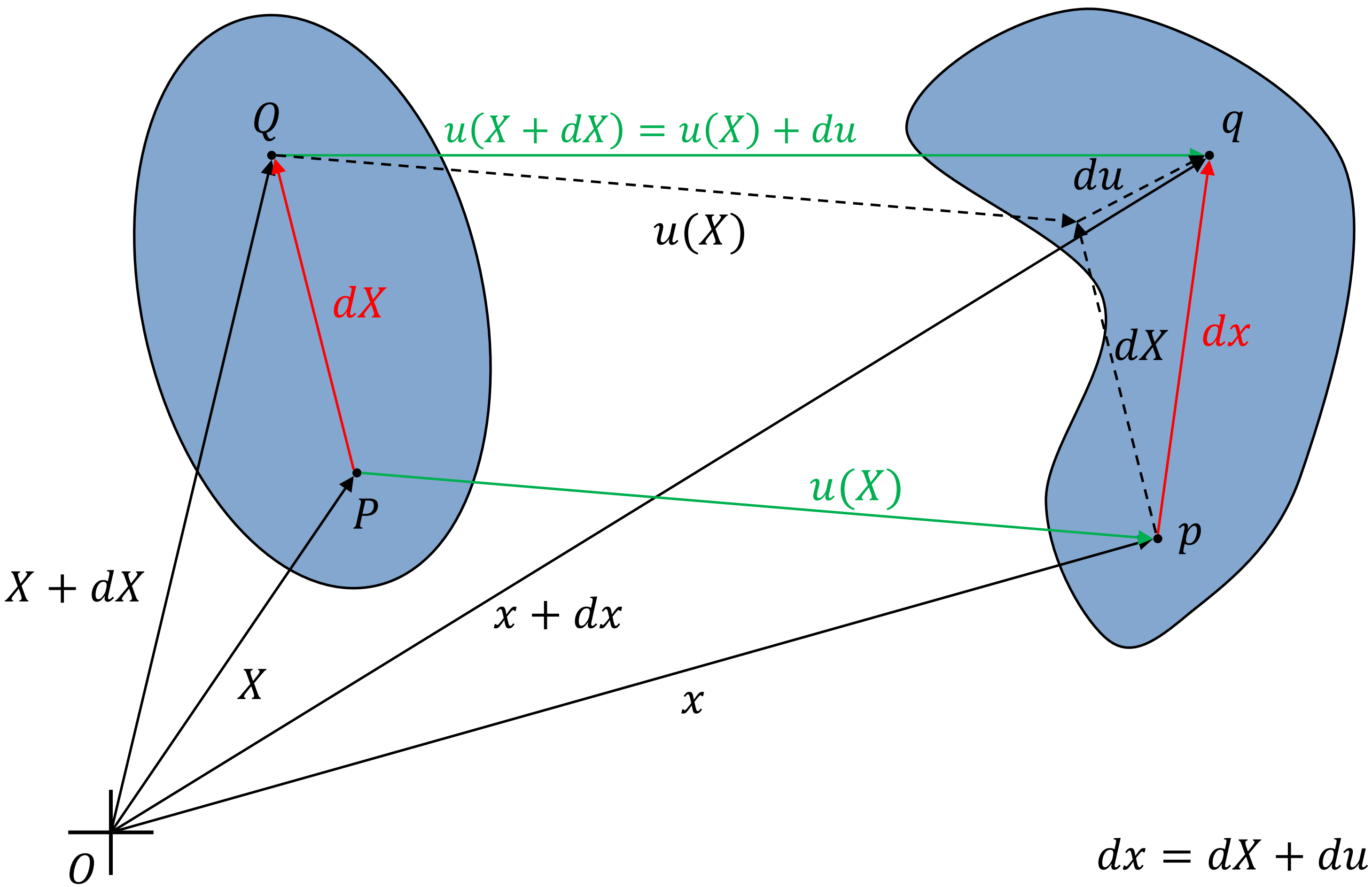
- A deformable object is defined by its rest shape and the material parameters
- In the discrete case, the object M is a discrete set of points with material coordinates $m \in M$ that samples the rest shape of the object
- When forces are applied, the object deforms
  - each m moves to a new location $x(m)$
  - $u(m) = x(m) - m$ can be seen as the displacement vector field
  - *e.g.* a constant displacement field is a translation of the object

Continuum mechanics

- Material coordinate P with position X is deformed to p with position x
- Material coordinate Q with position $X + dX$ is deformed to q with position $x + dx$
- If the deformation is very small (*i.e.* linear deformation in interval Δt), the displacements of the material coordinates can be described by

$$x + dx = X + dX + u(X + dX)$$
$$dx = X - x + dX + u(X + dX)$$
$$dx = dX + u(X + dX) - u(X)$$
$$dx = dX + du$$

Continuum mechanics



$$dx = dX + du$$

Continuum mechanics

- du is the relative displacement vector
- It represents the relative displacement of Q with respect to P in the deformed configuration
- Now if we assume that Q is very close to P and that the displacement field is continuous, we have

$$u(X + dX) = u(X) + du \approx u(X) + \nabla u * dX$$

where the gradient of the displacement field is (in 3D) the 3×3 matrix of the partial derivatives of u

$$\nabla u = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} \text{ where } u = (u, v, w)^T$$

Continuum mechanics

- With that definition of the relative displacement vector, we can calculate the relative position of q

$$dx = dX + du = dX + \nabla u * dX$$
$$dx = (I + \nabla u)dX = F * dX$$

- We call F the material deformation gradient tensor
- It characterizes the local deformation at a material coordinate, *i.e.* provides a mapping between the relative position at rest and the relative position after deformation

Strain and stress

- The strain and stress are related to the material deformation gradient tensor F, and so to the displacement field u
- In interactive applications, we usually use the Green-Cauchy strain tensors

$$\epsilon_G = \frac{1}{2}(\nabla u + (\nabla u)^T + (\nabla u)^T \nabla u)$$

$$\epsilon_C = \frac{1}{2}(\nabla u + (\nabla u)^T)$$

- And stress tensor from Hooke's linear material law

$$\sigma = E * \epsilon$$

where E is the elasticity tensor and depends on the Young's modulus and Poisson's ratio (and more)

Modeling soft bodies

- Two types of approaches are possible to simulate deformable models
  - Lagrangian methods (particle-based)
    - a model consists of a set of moving points carrying material properties
    - convenient to define an object as a connected mesh of points or a cloud of points, suitable for deformable soft bodies
    - examples: Finite Element/Difference/Volume methods, Mass-spring system, Coupled particle system, Smoothed particle hydrodynamics
  - Eulerian methods (grid-based)
    - scene is a stationary set of points where the material properties change over time
    - boundary of object not explicitly defined, suitable for fluids

Finite Element Method

- FEM is used to numerically solve partial differential equations (PDEs) by discretization of the volume into a large finite number of disjoint elements (3D volumetric mesh)

- The PDE of the equation of motion governing dynamic elastic materials is given by

$$\rho * a = \nabla \cdot \sigma + F$$

where ρ is the density of the material, a is the acceleration of the element, $\nabla \cdot \sigma$ is the divergence of stress (internal forces) and F the external forces

Finite Element Method

- First the deformation field u is estimated from the positions of the elements within the object
- Given the current local strain, the local stress is calculated
- The equation of motion of the element nodes is obtained by integrating the stress field over each element and relating this to the node accelerations through the deformation energy

$$E = \int_V \epsilon(m) * \sigma(m)\, dm$$

Finite Differences Method

- If the object M is sampled using a regular spatial grid, the PDE can be discretized using finite differences (FD)
  - easier to implement that FEM
  - difficult to approximate complex boundaries
- Deformation energy comes from difference between metric tensors of the deformed and original shapes
- Derivative of this energy is discretized using FD
- Finally semi-implicit integration is used to move forward through time

Finite Volume Method

- In the Finite Volume method, the nodal forces are not calculated from the derivation of the deformation energy
- But first internal forces f per unit area of a plane (of normal n) are calculated from the stress tensor

$$f = \sigma * n$$

- The total force acting on a face A of an element is

$$f_A = \int_A \sigma \, dA = A * \sigma * n$$

  for planar element faces (stress tensor constant within an element)
- By iterating on all faces of an element, we can then distribute (evenly) the force among adjacent nodes

Boundary Element Method

- The boundary element method simplifies the finite element method from a 3D volume problem to a 2D surface problem
  - PDE is given for boundary deformation
  - only works for homogenous material
  - topological changes more difficult to handle

Mass-Spring System

- An object consists of point masses connected by a network of massless springs
- The state of the system is defined by the positions x_i and velocities v_i of the masses $i = 1 \cdots n$
- The force f_i on each mass is computed from the external forces (*e.g.* gravity, friction) and the spring connections with its neighbors
- The motion of each mass point $f_i = m_i a_i$ is summed up for the entire system in

$$M * a = f(x, v)$$

where M is a $3n \times 3n$ diagonal matrix

Mass-Spring System

- The mass points are usually regularly spaced in a 3D lattice
- The 12 edges are connected by structural springs
  - resist longitudinal deformations
- Opposite corner mass points are connected by shear springs
  - resist shear deformations
- The rest lengths define the rest shape of the object

Mass-Spring System

- The force acting on mass point i generated by the spring connecting i and j is

$$f_i = Ks_i(|x_{ij}| - l_{ij})\frac{x_{ij}}{|x_{ij}|}$$

where x_{ij} is the vector from positions i to j, K_i is the stiffness of the spring and l_{ij} is the rest length

- To simulate dissipation of energy along the distance vector, a damping force is added

$$f_i = Kd_i\left(\frac{(v_j - v_i)^T x_{ij}}{x_{ij}{}^T x_{ij}}\right)x_{ij}$$

Mass-Spring System

- Intuitive system and simple to implement
- Not accurate as does not necessarily converge to correct solution
  - depends on the mesh resolution and topology
  - spring constants chosen arbitrarily
- Can be good enough for games, especially cloth animation
  - as can have strong stretching resistance and weak bending resistance

Coupled Particle System

- Particles interact with each other depending on their spatial relationship
- Referred to as spatially coupled particle system
  - these relationships are dynamic, so geometric and topological changes can take place
- Each particle p_i has a potential energy E_{Pi} which is the sum of the pairwise potential energies between the particle p_i and the other particles

$$E_{Pi} = \sum_{j \neq i} E_{Pij}$$

Coupled Particle System

- The force f_i applied on the particle at position p_i is

$$f_i = -\nabla_{p_i E_{Pi}} = -\sum_{j \neq i} \nabla_{p_i E_{Pij}}$$

where $\nabla_{p_i E_{Pi}} = \left(\frac{dE_{Pi}}{dx_i}, \frac{dE_{Pi}}{dy_i}, \frac{dE_{Pi}}{dz_i} \right)$

- To reduce computational costs, interactions to a neighborhood is used
  - potential energies weighted according to distance to particle

Smoothed Particle Hydrodynamics

- SPH uses discrete particles to compute approximate values of needed physical quantities and their spatial derivatives
  - obtained by a distance-weight sum of the relevant properties of all the particles which lie within the range of a smoothing kernel
- Reduces the programming and computational complexity
  - suitable for gaming applications

Smoothed Particle Hydrodynamics

- The equation for any quantity A at any point r is given by

$$A(r) = \sum_j m_j \frac{A_j}{\rho_j} W(|r - r_j|, h)$$

  - where W is the smoothing kernel (usually Gaussian function or cubic spline) and h the smoothing length (max influence distance)
  - for example the density can be calculated as

$$\rho(r) = \sum_j m_j W(|r - r_j|, h)$$

- It is applied to pressure and viscosity forces, while external forces are applied directly to the particles

Smoothed Particle Hydrodynamics

- The spatial derivative of a quantity can be calculated from the gradient of the kernel
  - the equations of motion are solved by deriving forces
- By varying automatically the smoothing length of individual particles you can tune the resolution of a simulation depending on local conditions
  - typically use a large length in low particle density regions and a smaller length in high density regions
- Easy to conserve mass (constant number of particles) but difficult to maintain incompressibility of the material

Eulerian Methods

- Eulerian methods are typically used to simulate fluids (liquids, smoke, lava, cloud, *etc.*)
- The scene is represented as a regular voxel grid, and fluid dynamics describes the displacements
  - we apply finite difference formulation on the voxel grid
  - the velocity is stored on the cell faces and the pressure is stored at the center of the cells
- Heavily rely on the Navier-Stokes equations of motion for a fluid

Navier-Stokes equations

- They represent the conservation of mass and momentum for an incompressible fluid

$$\nabla \cdot u = 0$$

$$\overbrace{\rho(\underbrace{u_t}_{\text{Unsteady acceleration}} + \underbrace{u \cdot \nabla u}_{\text{Convective acceleration}})}^{\text{Inertia (per volume)}} = \overbrace{\underbrace{\nabla \cdot (\nu \nabla u)}_{\text{Viscosity}} - \underbrace{\nabla p}_{\text{Pressure gradient}}}^{\text{Divergence of stress}} + \underbrace{f}_{\text{Other body forces}}$$

  - u_t is the time derivative of the fluid velocity (the unknown), p is the pressure field, ν is the kinematic viscosity, f is the body force per unit mass (usually just gravity ρg)

Navier-Stokes equations

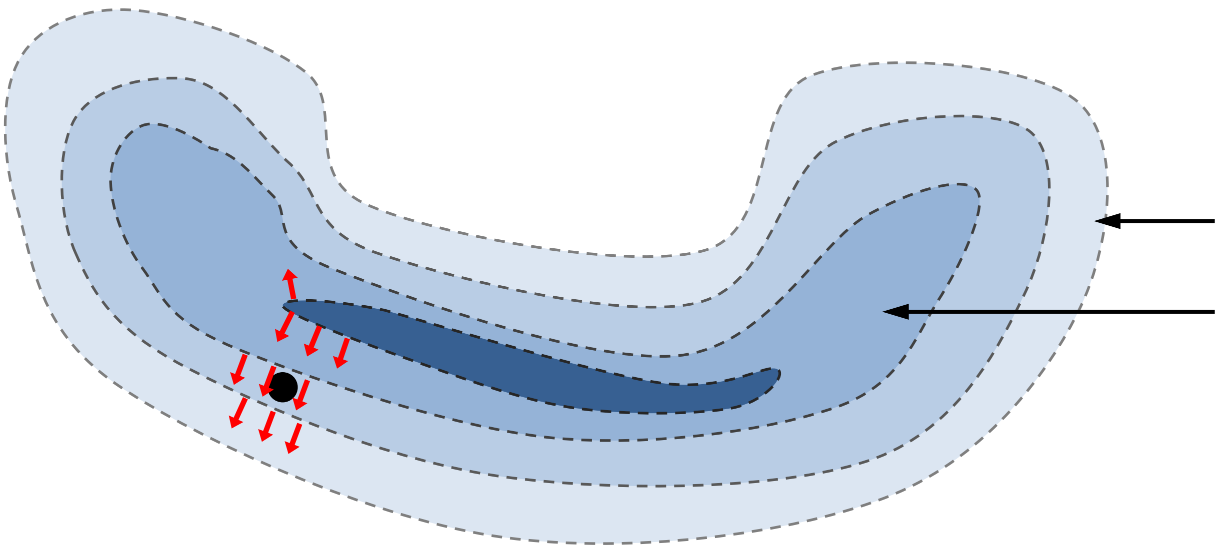
- First f is scaled by the time step and added to the current velocity
- Then the advection term $u \cdot \nabla u$ is solved
  - it governs how a quantity moves with the underlying velocity field (time independent, only spatial effect)
  - it ensures the conservation of momentum
  - sometimes called convection or transport
  - solved using a semi-Lagrangian technique

Navier-Stokes equations

- Then the viscosity term $\nabla \cdot (\nu \, \nabla u) = \nu \nabla^2 u$ is solved
  - it defines how a cell interchanges with its neighbors
  - also referred to as diffusion
  - viscous fluids can be achieved by applying diffusion to the velocity field
  - it can be solved for example by finite difference and an explicit formulation
    - 2-neighbor 1D:
      $u_i(t) = \nu * \Delta t * (u_{i+1} + u_{i-1} - 2u_i)$
    - 4-neighbor 2D:
      $u_{i,j}(t) = \nu * \Delta t * \left(u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}\right)$
    - Taking the limit gives indeed $\nu \nabla^2 u$

Navier-Stokes equations

- Finally, the pressure gradient is found so that the final velocity will conserve the volume (*i.e.* mass for incompressible fluid)
  - sometimes called pressure projection
  - it represents the resistance to compression $-\nabla p$



Navier-Stokes equations

- We make sure the velocity field stays divergence-free with the second equation $\nabla \cdot u = 0$, *i.e.* the velocity flux of all faces at each fluid cell is zero (everything that comes in, goes out)
- The equation $u(t + \Delta t) = u(t) - \Delta t \nabla p$ is solved from its combination with $\nabla \cdot u = 0$, giving

$$\nabla \cdot u(t + \Delta t) = \nabla \cdot u(t) - \Delta t \nabla \cdot (\nabla p) = 0$$
$$\Leftrightarrow \Delta t \nabla^2 p = \nabla \cdot u(t)$$

with which we solve for p, then plug back in the $u(t + \Delta t)$ equation to calculate the final velocity

Navier-Stokes equations

- Compressible fluids can also conserve mass, but their density must change to do so
- Pressure on boundary nodes
  - In free surface cells, the fluid can evolve freely ($p = 0$)
    - so that for example a fluid can splash into the air
  - Otherwise (*e.g.* in contact with a rigid body), the fluid cannot penetrate the body but can flow freely in tangential directions $u_{boundary} \cdot n = u_{body} \cdot n$

End of
Soft body physics

Next

Physics engine design and implementation